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Perturbed Characteristic Functions. III

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As an improvement on earlier work, it is shown how perturbations of characteristic functions can be obtained by a recursive procedure which avoids the necessity to find the perturbation of the extremal joining given terminal points, that is, all integrations now go along unperturbed extremals. For the sake of brevity, only the important case of the world characteristic is dealt with here.

1. INTRODUCTION

In an earlier paper (Buchdahl, 1985), hereafter referred to as II, I considered the first- and second-order perturbations of characteristic functions. What is to be understood by this is adequately set out in the Introduction to II. A generic expression for the total perturbation ΔV of V is given by equation (II.11), and this was the starting point for the determination of $V^{(1)}$ and $V^{(2)}$. (Except where otherwise indicated, the notation of II is retained here.) The trouble with this procedure is that the integrations which are part of it go along perturbed extremals. Thus, granted that everything required for the determination of $V^{(r-1)}$ is already known, to find $V^{(r)}$, one must first find the $q_{(r-1)}(u)$, $k=1,\ldots,n$. While these are implicit in $V^{(r-1)}$, already known, one is confronted, for all but the smallest values of r, with very tedious calculations. It is therefore desirable to find a procedure which avoids these vexatious complications.

To this end, a different starting point is adopted here, namely the Hamilton–Jacobi equations satisfied by V, one at each endpoint. Rather than consider the most general case, I confine my attention here to the world characteristic. (This was considered in some detail also in II.) Then, by almost trivial means, a recursive set of equations for the derivatives

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$dV^{(r)}/ds_0$ $(r = 1, 2, \ldots)$ is obtained in Section 2. These are derivatives along the unperturbed extremal $\mathscr{E}_0$ at any one of its points. In consequence, the need to find the perturbations of the extremals no longer arises. A situation of particular interest and simplicity is that of the flat unperturbed metric, for then the form of $V^{(0)}$ and the corresponding geodesics are immediately at hand, as briefly set out for later convenience in Section 3. Finally, in Section 4, a specific metric which already occurs in Section 5 of II is again used for illustrative purposes.

2. THE EQUATIONS FOR $dV^{(r)}/ds_0$ $(r=0, 1, 2, \ldots)$

The Hamilton-Jacobi equations satisfied by the world-characteristic $V(x'^1, x'^2, x'^3, x'^4; x^1, x^2, x^3, x^4)$ are, at the initial point P

$$g^{ij} V_{,i} V_{,j} = 1 \tag{2.1}$$

and, at the final point P',

$$g'^{ij} V_{,i'} V_{,j'} = 1 \tag{2.2}$$

where subscripts i and i' following a comma denote derivatives with respect to x^i and x'^i, respectively, as usual. [Note that as regards primed variables the notation differs from that of Synge (1960).] The perturbation of the metric is most conveniently represented by the equation

$$\bar{g}^{ij} = \sum_{s=0}^{\infty} \varepsilon^s \bar{g}^{ij}_{(s)}, \tag{2.3}$$

the bars indicating that one is concerned with general points, i.e., not merely with P. The parameter ε is supposed to be so small in absolute value that the series on the right of (2.3) converges. Likewise, it is to be taken for granted that the concomitant series for V, i.e.,

$$V = \sum_{s=0}^{\infty} \varepsilon^s V^{(s)} \tag{2.4}$$

converges.

Now insert (2.3) and (2.4) in (2.2), but suppose the final point to be notionally not necessarily P', but any point $\bar{P}$ on $\mathscr{E}_0$. Then, if ε be regarded as freely variable, the factor multiplying ε^r in the resulting equation must vanish for all values of $r \geq 1$, so that one has

$$\sum_{a=0}^{r} \sum_{b=0}^{a} \bar{g}^{ij}_{(b)} \bar{V}^{(r-a)}_{,i} \bar{V}^{(a-b)}_{,j} = 0 \qquad (r \geq 1) \tag{2.5}$$

(Here $\bar{V}$ of course stands for V with x'^i replaced by $\bar{x}^i$, $i = 1, \ldots, 4$, and the derivatives are with respect to $\bar{x}^i$, $\bar{x}^j$.) Next, separate out the terms that have $a = b = 0$:

$$\bar{g}_0^{ij}\bar{V}_{,i}^{(0)}\bar{V}_{,j}^{(r)} + \sum_{a=1}^{r-1}\bar{g}_{(0)}^{ij}\bar{V}_{,i}^{(r-a)}\bar{V}_{,j}^{(a)} + \sum_{a=1}^{r}\sum_{b=1}^{a}\bar{g}_{(b)}^{ij}\bar{V}_{,i}^{(r-a)}\bar{V}_{,j}^{(a-b)} = 0 \quad (2.6)$$

Now,

$$\bar{g}_{(0)}^{ij}\bar{V}_{,i}^{(0)} = d\bar{x}^j/ds_0$$

so that, along $\mathscr{C}_0$,

$$d\bar{V}^{(r)}/ds_0 = \bar{V}_{,j}^{(r)}\,dx^j/ds_0 = \bar{g}_{(0)}^{ij}\bar{V}_{,i}^{(0)}\bar{V}_{,j}^{(0)} \quad (2.7)$$

Therefore, from (2.6), after integration,

$$\bar{V}^{(r)} = -\frac{1}{2}\int_{P'}^{P'}\left\{\sum_{a=1}^{r-1}\bar{g}_{(0)}^{ij}\bar{V}_{,i}^{(r-a)}\bar{V}_{,j}^{(a)} + \sum_{a=1}^{r}\sum_{b=1}^{a}\bar{g}_{(b)}^{ij}\bar{V}_{,i}^{(r-a)}\bar{V}_{,j}^{(a-b)}\right\}ds_0 \quad (2.8)$$

This is the equation from which $V^{(1)}, V^{(2)}, \ldots$, may be found in turn. For later use it is convenient to write (2.8) out in full for $r = 1, 2, 3$:

$$V^{(1)} = -\int_P^{P'} \tfrac{1}{2}\bar{g}_{(1)}^{ij}\bar{V}_{,i}^{(0)}\bar{V}_{,j'}^{(0)}\,ds_0 \quad (2.9)$$

$$V^{(2)} = -\int_P^{P'}\{\tfrac{1}{2}g_{(0)}^{ij}\bar{V}_{,i}^{(1)}\bar{V}_{,j'}^{(1)} + \bar{g}_{(1)}^{ij}\bar{V}_{,i}^{(0)}\bar{V}_{,j}^{(1)} + \tfrac{1}{2}\bar{g}_{(2)}^{ij}\bar{V}_{,i}^{(0)}\bar{V}_{,j}^{(0)}\}\,ds_0 \quad (2.10)$$

$$V^{(3)} = -\int\{\bar{g}_{(0)}^{ij}\bar{V}_{,i}^{(1)}\bar{V}_{,j}^{(2)} + \bar{g}_{(1)}^{ij}(\bar{V}_{,i}^{(0)}\bar{V}_{,j}^{(2)} + \tfrac{1}{2}\bar{V}_{,i}^{(1)}\bar{V}_{,j}^{(1)})$$
$$+\bar{g}_{(2)}^{ij}\bar{V}_{,i}^{(0)}\bar{V}_{,j}^{(1)} + \tfrac{1}{2}\bar{g}_{(3)}^{ij}\bar{V}_{,i}^{(0)}\bar{V}_{,j}^{(0)}\}\,ds_0 \quad (2.11)$$

3. FLAT UNPERTURBED METRIC

A situation of particular interest is that in which the unperturbed metric is that of a flat space, so that, with an appropriate choice of coordinates, $g_{(0)}^{ij} = \eta^{ij} := \mathrm{diag}(1, 1, 1, -1)$. In this case $V^{(0)}$ is simply the four-dimensional distance l between P and P', i.e.,

$$V^{(0)} = l := (\eta_{ij}\xi^i\xi^j)^{1/2} \quad (3.1)$$

where $\xi^i := x'^i - x^i$. Moreover, on $\mathscr{C}_0$ points $\bar{P}$, with coordinates $\bar{x}^i$, may be parametrized as follows:

$$\bar{x}^i = \lambda\xi^i + x^i \quad (3.2)$$

($\xi^i := x^{\prime i} - x^i$), the range of λ being (0, 1). Evidently the distance between P and $\bar{P}$ is $\bar{l} = \lambda l$, and $ds_0 = l\,d\lambda$. Also,

$$\bar{V}^{(0)}_{,i} = l^{-1}\xi_i \tag{3.3}$$

with the convention that indices are moved exclusively with the unperturbed metric. Thus, to lowest order,

$$V^{(1)} = -\tfrac{1}{2}l^{-1}\xi_i\xi_j \int_0^1 \bar{g}^{ij}_{(1)}\,d\lambda \tag{3.4}$$

When, at least to this order, spacetime is conformally flat, i.e., there exists a scalar function ϕ such that

$$g^{ij}_{(1)} = \eta^{ij}\phi \tag{3.5}$$

then

$$V^{(1)} = -\tfrac{1}{2}l \int_0^1 \bar{\phi}\,d\lambda \tag{3.6}$$

4. EXPLICIT EXAMPLE

The relative simplicity of the present approach may be illustrated by reconsidering the example chosen in Section 5 of II, namely,

$$g^{ij} = (1 + \varepsilon x^1)^2\eta^{ij} \tag{4.1}$$

Evidently the x^1 coordinate occupies a privileged position, and the ubiquitous appearance of the index 1 is a nuisance. It may therefore simply be omitted whenever this is not likely to lead to confusion; e.g., $x := x^1$, $\xi := \xi^1$. Then

$$\bar{g}^{ij}_{(1)} = 2\eta^{ij}\bar{x}, \qquad \bar{g}^{ij}_{(2)} = \eta^{ij}\bar{x}^2, \qquad \bar{g}^{ij}_{(r)} = 0 \qquad (r > 2) \tag{4.2}$$

Now, with $\bar{\phi} = 2\bar{x} = 2\lambda\xi + 2x$, (3.6) immediately gives

$$V^{(1)} = -\tfrac{1}{2}l(x' + x) \tag{4.3}$$

Next, one has $\bar{V}^{(1)} = -\frac{1}{2}\bar{l}(\bar{x} + x)$, whence

$$\bar{V}^{(1)}_{,i} = -\tfrac{1}{2}\{l^{-1}(\lambda\xi + 2x)\xi_i + \lambda l\delta^1_i\} \tag{4.4}$$

since, after differentiation with respect to $\bar{x}^i$, equation (3.2) and its concomitants may be used. With (3.3), (4.2), and (4.4), the integrand, I_2, say, of the integral on the right of (2.10) becomes

$$I_2 = -\tfrac{1}{8}(9\xi^2\lambda^2 + 16\xi x\lambda + 8x^2 - l^2\lambda^2) \tag{4.5}$$

Then

$$V^{(2)} = -l \int_0^1 I_2 \, d\lambda = \tfrac{1}{8} l (3\xi^2 + 8\xi x + 8x^2 - \tfrac{1}{3} l^2) \tag{4.6}$$

or

$$V^{(2)} = \tfrac{1}{8} l [(3x'^2 + 2x'x + 3x^2) - \tfrac{1}{3} l^2] \tag{4.7}$$

a result in harmony with equation (II.41); but even in this low order it has emerged considerably more easily than it did in II. By the same token, progression to the third order is now quite straightforward. Thus, $\bar{V}^{(2)}$ follows from (4.7) by replacing x' by $\bar{x}$ and l by $\bar{l}$, as usual; and then

$$\bar{V}^{(2)}_{,i} = \tfrac{1}{8}\{l^{-1}(3\lambda^2\xi^2 + 8\lambda x\xi + 8x^2 - \lambda^2 l^2) + 2\lambda l(3\lambda\xi + 4x)\delta_i^1\} \tag{4.8}$$

on ε_0. In view of (3.3), (4.4), (4.8), and (4.2), everything in the integrand, I_3, say, of the integral in (2.10) is now known explicitly. Substituting these expressions in I_3, a little elementary algebra shows that explicitly

$$I_3 = \tfrac{1}{8}\{(10\lambda^3\xi^3 + 27\lambda^2\xi^2 x + 24\lambda\xi x^2 + 8x^3) - \lambda^2 l^2(2\lambda\xi + 3x)\} \tag{4.9}$$

Therefore

$$V^{(3)} = -l \int_0^1 I_3 \, d\lambda = -\tfrac{1}{8} l \{(\tfrac{5}{2}\xi^3 + 9\xi^2 x + 12\xi x^2 + 8x^3) - l^2(\tfrac{1}{2}\xi + x)\} \tag{4.10}$$

which may also be written

$$V^{(3)} = \tfrac{1}{16} l (x' + x)[l^2 - (5x'^2 - 2x'x + 5x^2)] \tag{4.11}$$

This result is in harmony with the known closed form (II.43) of V.

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